

Propagation in Polymer Parameterised Field Theory

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The Hamiltonian constraint operator in Loop Quantum Gravity acts ultralocally. Smolin has argued that this ultralocality seems incompatible with the existence of a quantum dynamics which propagates perturbations between macroscopically separated regions of quantum geometry. We present evidence to the contrary within an LQG type ‘polymer’ quantization of two dimensional Parameterised Field Theory (PFT). PFT is a generally covariant reformulation of free field propagation on flat spacetime. We show explicitly that while, as in LQG, the Hamiltonian constraint operator in PFT acts ultralocally, states in the joint kernel of the Hamiltonian and diffeomorphism constraints of PFT *necessarily* describe propagation effects. The particular structure of the finite triangulation Hamiltonian constraint operator plays a crucial role, as does the necessity of imposing (the continuum limit of) its kinematic adjoint as a constraint. Propagation is seen as a property encoded by *physical* states in the kernel of the constraints rather than that of repeated actions of the finite triangulation Hamiltonian constraint on kinematic states. The analysis yields robust structural lessons for putative constructions of the Hamiltonian constraint in LQG for which ultralocal action co-exists with a description of propagation effects by physical states.

I. INTRODUCTION

A key open problem in Loop Quantum Gravity is a satisfactory definition of the Hamiltonian constraint operator which generates its quantum dynamics. A mathematically well defined procedure to construct this operator using canonical quantization techniques was developed by Thiemann [1]¹. While significant progress has been made since Thiemann’s pioneering work the construction still yields an operator which is far from unique. Therefore it is desirable to subject candidate operators to further requirements so as to cut down on the quantization choices responsible for this non-uniqueness. For example, in [4] a Hilbert space is constructed which supports the action of the operator so that the operator may be confronted with adjointness properties. Another example is of recent work which attempts to impose the requirement that the candidate operator yield an anomaly free algebra of constraints so as to impose spacetime covariance in the quantum theory [5–7].

Here we are interested in yet another requirement, namely that the quantum dynamics describe the propagation of perturbations from one part of the quantum geometry to another. This requirement was articulated several years ago by Smolin [8]. In his work, Smolin also offers a critique of Thiemann’s general construction and concludes that such a construction method yields a constraint action which is ‘too local’ to allow for propagation effects in the

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¹ This work itself was based on cumulative progress achieved by many workers; for a detailed bibliography see, for e.g. the review article [2] and the book [3]

quantum theory. More in detail, recall that quantum states in LQG are superpositions of ‘spin network’ states labelled by graphs. Each such state describes 1d excitations of spatial geometry along the edges of the graph which underlies the state. All known versions of the Hamiltonian constraint which derive from the general Thiemann procedure act only at vertices of the graph and the action at one vertex is independent of the action at neighbouring vertices. Further, repeated actions of the Hamiltonian constraint create a finer and finer nested structure about each vertex. Thus, modifications of the graph structure wrought by the action of the Hamiltonian constraint in the immediate vicinity of one vertex do not seem to propagate to other vertices. Since such propagation is thought of as the non-perturbative seed for graviton propagation in semiclassical states in LQG, one would like the quantum dynamics to allow for such propagation. Since the issue of the (semi)classical limit of LQG is still open, Smolin’s criticisms are based, at least partly, on physical intuition. Nevertheless, these criticisms seem compelling and this article seeks to address them.

Given the complications of full blown gravity and the open issues within LQG, we focus on Smolin’s criticisms in the context of a generally covariant field theoretic toy model which has already proven to be extremely useful in addressing other issues concerned with the Hamiltonian constraint. The model is that of 1+1 Parameterised Field Theory (PFT) which is a generally covariant reformulation of free massless scalar field theory on 1+1 Minkowski spacetime in which, in addition to the scalar field, the Minkowskian coordinates are treated as dynamical variables to be varied in the action. Since on the one hand the classical theory describes scalar wave *propagation*, and on the other, a complete LQG type ‘polymer’ quantization exists for the model [9, 10, 13] this model serves as an ideal testing ground for an analysis and possible resolution of Smolin’s criticisms. By virtue of its general covariance, the dynamics of PFT is driven by Hamiltonian and diffeomorphism constraints. The kinematic Hilbert space is spanned by ‘charge network states’. Each such state is labelled by a 1d graph on the Cauchy slice, the edges of the graph being colored by integer value charges associated with quantum excitations of the Minkowskian coordinates and the scalar field. The physical Hilbert space can be constructed by group averaging techniques [11] and, despite being a rigorous quantization of *continuum* scalar field theory, these physical states describe quantum scalar field excitations on *discrete* spacetime. More in detail, a superselection sector of such states exists which describes quantum scalar field excitations *propagating* on a discrete spacetime *lattice*, the lattice spacing being governed by the analog of the Barbero-Immirzi parameter [12]. For reasons explained in [10] we call this sector the ‘finest lattice’ sector. The Hamiltonian constraint operator can also be constructed following the broad procedure introduced by Thiemann and quantization choices can be made in such a way that its action is an infinitesimal version of the finite transformations used in the group averaging procedure referred to above (see [13] and Section IV). The Hamiltonian constraint operator so constructed acts ultralocally. By this we mean that, as in LQG, the finite triangulation Hamiltonian constraint ² acts on vertices of kinematical states, its action at a vertex is only sensitive to structure in a small neighbourhood of that vertex and its action at one vertex is independent of that at other vertices. As a result, its repeated action at a vertex does

² Recall that the Thiemann procedure introduces a set of finer and finer triangulations of the Cauchy slice, chooses at each triangulation an approximant to the constraint constructed from the basic holonomy-flux functions of the theory so that at infinitely fine triangulation the approximant becomes exact, replaces the classical functions by quantum operators, and, finally, defines the resulting operator in a limit of infinitely fine triangulation.

not lead by itself to any propagation for reasons similar to that articulated by Smolin in the context of LQG.

Nevertheless, as we show in this work, despite this ultralocal action, the joint kernel of the Hamiltonian constraint operator and the diffeomorphism constraint operator necessarily contains the finest lattice sector of physical states which *do* provide a description of propagation effects. Thus our work shows that propagation is *not* to be seen as a property of repeated actions of the (finite triangulation) Hamiltonian constraint but rather as a (logically independent) property encoded in physical states which are in the joint kernel of the Hamiltonian and diffeomorphism constraints. With this shift in focus to the structure of physical states in the kernel of the constraints, the two key features which enable propagation effects turn out to be

- (i) the structure of the Hamiltonian constraint at finite triangulation which is quite different from the structure of Thiemann's Hamiltonian constraint despite their shared property of ultralocality.
- (ii) the imposition of the continuum limit of the finite triangulation Hamiltonian constraint as well as the continuum limit of the adjoint of this finite triangulation constraint, as operator constraints for physical states.

We believe that the emphasis on propagation as a property encoded in physical states together with the general structural lessons learnt from PFT are robust and applicable to LQG and offer a way out of Smolin's criticisms. In other words, while Smolin's criticisms seem to hold for Thiemann's choice of Hamiltonian constraint and while this choice leads to an ultralocal action, it does not follow that the same criticisms need hold for other choices even if these choices also lead to an ultralocal action. The obstacle to propagation is then *not* ultralocality. Hence, while the Thiemann procedure does seem to lead to ultralocal constraint action, we are optimistic that there exist choices of constraints constructed via the general Thiemann procedure which co-exist with physical states describing propagation effects.³

The layout of this paper is as follows. In section II we provide a brief review of classical PFT and its polymer quantization wherein physical states are constructed through group averaging techniques. The interested reader may consult [9, 10] for details of the formalism. In section III we focus on a particular physically relevant superselection sector of the Hilbert space known as the 'finest lattice sector'. We review the pictorial interpretation of kinematic and group averaged physical states in this sector in terms of discrete slices on a spacetime lattice carrying quantum matter excitations. In section IV we review the action of the Hamiltonian constraint from [13] and show that its action is ultralocal and that its repeated action on a finest lattice charge net does not cause long range propagation exactly as anticipated by Smolin. In the pictorial interpretation of section III this repeated action fails to evolve the discrete Cauchy slice and its data beyond a single lattice step. It turns out that the key to long range propagation is getting beyond one lattice step to two lat-

³ Preliminary work [14] on the ' $U(1)^3$ ' model [6, 7, 15] indicates that the notion of physical states should be further restricted by demanding that they lie not only in the kernel of the Hamiltonian and diffeomorphism constraints but also in that (of certain combinations of the Hamiltonian and) the 'electric' diffeomorphism constraints, the latter being obtained by smearing the diffeomorphism constraint by triad dependent shifts [6]. It is not clear to us if this further restriction in the context of Thiemann's specific choices in [1] suffices to yield physical propagation effects or if a different choice of the Hamiltonian constraint is required in addition to this restriction; at the moment we believe the latter is more likely.

tice steps. In section V we show how elements in the joint kernel of the Hamiltonian and diffeomorphism constraints encode evolution beyond a single lattice step to a second lattice step. We are able to isolate the key structural properties (see (i) and (ii) above) of the Hamiltonian constraint responsible for this. This is the main result of this paper. Once we have demonstrated evolution beyond a single lattice step to two lattice steps, a technical proof can be constructed to show that long range evolution is encoded in physical states. We relegate this proof to the appendix as it relies on certain detailed technicalities discussed in [9, 10] for which familiarity is assumed. Section VI contains a discussion of our results in the context of LQG.

II. REVIEW OF POLYMER PFT

A. Classical Theory

The action for a free scalar field f on a fixed flat 2 dimensional spacetime (M, η_{AB}) in terms of global inertial coordinates X^A , $A = 0, 1$ is

$$S_0[f] = -\frac{1}{2} \int_M d^2 X \eta^{AB} \partial_A f \partial_B f, \quad (2.1)$$

where the Minkowski metric in inertial coordinates, η_{AB} , is diagonal with entries $(-1, 1)$. As is well known, solutions to the equations ensuing from the action (2.1) take the form $f = f^+(X+T) + f^-(T-X)$ where f^\pm are arbitrary functions of their ‘light cone’ arguments. Due to this functional dependence f^+ describes left moving modes and f^- , right moving modes on the flat spacetime.

If, in the action (2.1), we use coordinates x^α , $\alpha = 0, 1$ (so that X^A are ‘parameterized’ by x^α , $X^A = X^A(x^\alpha)$), we have

$$S_0[f] = -\frac{1}{2} \int_M d^2 x \sqrt{\eta} \eta^{\alpha\beta} \partial_\alpha f \partial_\beta f, \quad (2.2)$$

where $\eta_{\alpha\beta} = \eta_{AB} \partial_\alpha X^A \partial_\beta X^B$ and η denotes the determinant of $\eta_{\alpha\beta}$. The action for PFT is obtained by considering the right hand side of (2.2) as a functional, not only of f , but also of $X^A(x)$ i.e. $X^A(x)$ are considered as 2 new scalar fields to be varied in the action so that $\eta_{\alpha\beta}$ is considered to be a functional of $X^A(x)$. Thus

$$S_{PFT}[f, X^A] = -\frac{1}{2} \int_M d^2 x \sqrt{\eta(X)} \eta^{\alpha\beta}(X) \partial_\alpha f \partial_\beta f. \quad (2.3)$$

Note that S_{PFT} is a diffeomorphism invariant functional of the scalar fields $f(x), X^A(x)$. Variation of f yields the equation of motion $\partial_\alpha(\sqrt{\eta} \eta^{\alpha\beta} \partial_\beta f) = 0$, which is just the flat space-time equation $\eta^{AB} \partial_A \partial_B f = 0$ written in the coordinates x^α . On varying X^A , one obtains equations which are satisfied if $\eta^{AB} \partial_A \partial_B f = 0$. This implies that $X^A(x)$ are undetermined functions (subject to the condition that the determinant of $\partial_\alpha X^A$ is non- vanishing). This 2 functions- worth of gauge is a reflection of the 2 dimensional diffeomorphism invariance of S_{PFT} . Clearly the dynamical content of S_{PFT} is the same as that of S_0 ; it is only that the diffeomorphism invariance of S_{PFT} naturally allows a description of the standard free field dynamics dictated by S_0 on *arbitrary* foliations of the fixed flat spacetime.

The spacetime is assumed to be of topology $\Sigma \times R$. A 1+1 Hamiltonian decomposition yields a phase space coordinatized by the canonically conjugate pairs $(T(x), \Pi_T(x)), (X(x), \Pi_X(x)), (f(x), \pi_f(x))$ where x coordinatizes the 1 dimensional $t = \text{constant}$ Cauchy slice Σ . For each value of x , the functions $(T(x), X(x))$ locate a point in flat spacetime by virtue of their interpretation as Minkowskian coordinates so that as x varies, $(T(x), X(x))$ describe an *embedding* of the Cauchy slice coordinatized by x into the flat spacetime. As a result we refer to the sector of phase space coordinatized by $(T(x), \Pi_T(x)), (X(x), \Pi_X(x))$ as the embedding sector. A canonical transformation can be made into ‘left and right moving’ embedding variables $(X^+, \Pi_+), (X^-, \Pi_-)$ with $X^\pm = T \pm X$ and Π_\pm their conjugate momenta. It is also useful to transform to the variables $Y^\pm = \pi_f + f'$ in the matter sector. It can be checked that the ‘+’ and ‘-’ variables Poisson commute with each other.

The dynamics is generated by a pair of constraints

$$H_\pm(x) = [\Pi_\pm(x) X^{\pm'}(x) \pm \frac{1}{4}(Y^\pm(x))^2]. \quad (2.4)$$

These constraints are of density weight two. In 1 spatial dimension their transformation properties under coordinate transformations are identical to those of spatial covector fields. Integrating them against multipliers N_\pm , which can therefore be thought of as spatial vector fields, one finds that the integrated ‘+’ (respectively ‘-’) constraint generates spatial diffeomorphisms on the ‘+’ (respectively ‘-’) variables while keeping the ‘-’ (respectively ‘+’) variables untouched. Thus, PFT dynamics can be thought of as the action of two *independent* spatial diffeomorphisms Φ^+, Φ^- on the ‘+’ and ‘-’ sectors of the phase space.

Instead of the H_\pm constraints we may consider the constraints

$$C_{diff} = H_+ + H_- \quad (2.5)$$

and

$$C_{ham} = H_+ - H_- \quad (2.6)$$

It can be checked that the diffeomorphism constraint C_{diff} generates spatial diffeomorphisms on the Cauchy data $(X^-, \Pi_-, X^+, \Pi_+, Y^+, Y^-)$ whereas the (density weight two) Hamiltonian constraint C_{ham} generates evolution of this data along the unit timelike normal n^α to the slice [16] (recall that the phase space data (X^+, X^-) define an embedded Cauchy slice in Minkowski spacetime; n^α is the unit timelike normal to this slice with respect to the flat spacetime metric).

In terms of the finite transformations Φ_+, Φ_- , it follows from (2.5), (2.6) that finite spatial diffeomorphisms generated by C_{diff} correspond to the choice $\Phi_+ = \Phi_-$ whereas finite transformations generated by the Hamiltonian constraint C_{ham} correspond to the choice $\Phi_+ = (\Phi_-)^{-1}$.

The relation between evolution on phase space data generated by the constraints and free field evolution of f on flat spacetime can be seen as follows [16]. The constraints (C_{diff}, C_{ham}) or, equivalently, (H_+, H_-) generate transformations of the phase space data $(X^\pm(x), Y^\pm(x))$ to new data $(X_{new}^\pm(x), Y_{new}^\pm(x))$. The phase space data $(X^+(x), X^-(x))$ define an embedded slice in flat spacetime. Initial data on this slice for evolution via the scalar wave equation can be given in terms of left and right moving values of the scalar field on the slice (or, ignoring issues of zero modes, the values of their derivatives). As discussed in [16] the relationship

between the phase space data Y^\pm and these derivatives is given by $\frac{\partial f}{\partial X^\pm}|_{X^\pm=X^\pm(x)} = \frac{Y^\pm(x)}{X^{\pm'}(x)}$. The transformed embedding data X_{new}^\pm then define an ‘evolved’ slice in flat spacetime with matter phase space data $X_{new}^{\pm'}(x)\frac{\partial f}{\partial X^\pm}|_{X^\pm=X_{new}^\pm(x)} = Y_{new}^\pm(x)$ where f is the restriction to the new slice of the solution to the wave equation with initial data $\frac{\partial f}{\partial X^\pm}$ on the old slice.

As in the previous works [9, 10, 13] we shall restrict attention to a flat spacetime (M, η) with cylindrical topology $S^1 \times R$. We denote the length of the $T = 0$ circle in the flat spacetime (M, η) by L . The cylindrical topology of M implies that any Cauchy slice Σ is circular. Certain subtleties related to the use of a single angular inertial coordinate X on the flat spacetime M as well as a single angular coordinate x on the Cauchy slice Σ arise but these subtleties and their ramifications constitute technical details which may be ignored in as much as the key arguments in this work are concerned. The interested reader may consult [16] and section IIC of [9] for an account of these subtleties in classical theory.

B. Quantum Theory

We shall concentrate mainly on the embedding sector in this brief review. For further details regarding the matter sector please see [10, 13]. We shall mention, but not explain in any detail, the subtleties in the quantum theory concerning the circular spatial topology because such details will only serve to distract from the main arguments in subsequent sections. The interested reader may consult [10] for such details.

1. Kinematics

The embedding sector Hilbert space is a tensor product of ‘+’ and ‘-’ sectors. On the ‘+’ sector the operator correspondents of functions on the ‘-’ sector of phase space act trivially and vice versa.

The ‘+’ embedding sector is spanned by an orthonormal basis of charge network states each of which is denoted by $|\gamma_+, \vec{k}^+\rangle$ where γ_+ is a graph i.e. a set of edges which cover the circle with each edge e labelled by a ‘charge’ k_e^+ , the collection of such charges for all the edges in the graph being denoted by \vec{k}^+ . The + embedding sector Hilbert space provides a representation of the Poisson bracket algebra between the classical ‘holonomy functions’ $\exp i(\sum_e (k_e^+ \int_e \Pi_+))$, and the embedding coordinate $X^+(x)$. In this representation the embedding momenta are polymerized so that the holonomy functions are well defined operators but the embedding momenta themselves are not. The embedding coordinate operator $\hat{X}^+(x)$ is well defined and the charge net states are eigen states of this operator. In particular the action of the embedding coordinate operators $\hat{X}^+(x)$ on a charge network state $|\gamma_+, \vec{k}^+\rangle$ when the argument x lies in the interior of an edge e of γ_+ is:

$$\hat{X}^+(x)|\gamma_+, \vec{k}^+\rangle = \hbar k_e^+ |\gamma_+, \vec{k}^+\rangle \quad (2.7)$$

Identical results hold for $+\rightarrow -$. The tensor product states $|\gamma_+, \vec{k}^+\rangle \otimes |\gamma_-, \vec{k}^-\rangle$ form a basis of the embedding sector Hilbert space and are referred to as embedding charge network states. By going to a graph finer than γ_+, γ_- , each such state can be equally well denoted by $|\gamma, \vec{k}^+, \vec{k}^-\rangle$ where each edge e of the graph γ is labelled by a *pair* of charges (k_e^+, k_e^-) and the collection of these charges is denoted by (\vec{k}^+, \vec{k}^-) . Such a state is an eigen ket of both

the \hat{X}^+ and \hat{X}^- operators. Similar to (2.7) the action of $\hat{X}^\pm(x)$ on the charge net $|\gamma, \vec{k}^+, \vec{k}^-\rangle$ when x is in the interior of an edge e of γ is:

$$\hat{X}^\pm(x)|\gamma, \vec{k}^+, \vec{k}^-\rangle = \hbar k_e^\pm |\gamma, \vec{k}^+, \vec{k}^-\rangle \quad (2.8)$$

The charges are chosen to be integer valued multiples of a dimensionful parameter $\frac{a}{\hbar}$ so that

$$\hbar k_e^\pm \in \mathbf{Z}a \quad (2.9)$$

where a is a Barbero- Immirzi like parameter. In the context of the circular spatial topology relevant to this work, we restrict attention, as in [10], to a value of a such that

$$\frac{L}{a} = N, \quad (2.10)$$

N being a positive integer.

The matter sector Hilbert space is also a tensor product of ‘+’ and ‘-’ sectors. On the ‘+’ sector the field Y^+ is polymerised and on the ‘-’ sector the Y^- field is polymerised. Thus taken together, neither Y^+ nor Y^- (and hence neither π_f nor f) exist as well defined operators. The ‘+’ sector provides a representation of matter holonomy functions $\exp i(\sum_e l_e^+ \int_e Y_+)$ on a basis of ‘+’ matter charge nets, each such charge net denoted by $|\gamma_+, \vec{l}^+\rangle$ in obvious notation. The matter charges \vec{l}^+ on each such charge net are real and are subject to the following restriction [10]: for every pair of edges $e, e' \in \gamma_+$, the matter charge difference $l_e^+ - l_{e'}^+$ is an integer valued multiple of a dimensionful parameter ϵ .⁴ The ‘-’ sector structure is identical. The parameter ϵ along with the parameter ‘ a ’ for the embedding sector constitute ‘Barbero-Immirzi’ parameters and label inequivalent representations. The tensor product states $|\gamma_+, \vec{l}^+\rangle \otimes |\gamma_-, \vec{l}^-\rangle$ form an orthonormal basis of the matter Hilbert space and are referred to as matter charge net states. By going to a fine enough graph any such state can be denoted, in notation similar to that for embedding states, as $|\gamma, \vec{l}^+, \vec{l}^-\rangle$.

The tensor product of the matter and embedding Hilbert spaces yields the kinematic Hilbert space \mathcal{H}_{kin} for PFT. This Hilbert space is spanned by charge net states each of which is a tensor product of a matter charge net and an embedding charge net. By going to a fine enough graph underlying the matter and embedding charge nets we may denote such a tensor product state by $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$.

Since the ‘+’ and ‘-’ sectors are independent, we also have the tensor product decomposition:

$$|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle = |\gamma_+, \vec{k}^+, \vec{l}^+\rangle \otimes |\gamma_-, \vec{k}^-, \vec{l}^-\rangle \quad (2.11)$$

where $|\gamma_\pm, \vec{k}^\pm, \vec{l}^\pm\rangle$ is itself a product of a ‘ \pm ’ embedding charge network and a ‘ \pm ’ matter charge network.

⁴ We do not impose the vanishing zero constraint [9, 10]. We remark further on the zero mode issue in section VI.

2. Gauge transformations generated by the constraints.

Recall that the finite transformations generated by the constraints H_+, H_- correspond to a pair of diffeomorphisms Φ_+, Φ_- . The quantum kinematics supports a unitary representation of these diffeomorphisms by the unitary operators $\hat{U}_+(\Phi_+), \hat{U}_-(\Phi_-)$. The operator $\hat{U}_+(\Phi_+)$ acts on a ‘+’ charge network state $|\gamma_+, k^{\vec{+}}, l^{\vec{+}}\rangle$ by moving the graph and its colored edges by the diffeomorphism Φ_+ while acting as identity on ‘-’ charge network states, and a similar action holds for $+ \rightarrow -$.⁵ We denote this action by

$$\hat{U}_\pm(\Phi_\pm)|\gamma_\pm, k^{\vec{\pm}}, l^{\vec{\pm}}\rangle =: |\gamma_{\pm, \Phi_\pm}, k_{\Phi_\pm}^{\vec{\pm}}, l_{\Phi_\pm}^{\vec{\pm}}\rangle \quad (2.12)$$

The action of finite gauge transformations on a charge net state $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ can then be deduced from equation (2.11):

$$\hat{U}_+(\Phi_+)\hat{U}_-(\Phi_-)|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle = |\gamma_{+, \Phi_+}, k_{\Phi_+}^{\vec{+}}, l_{\Phi_+}^{\vec{+}}\rangle \otimes |\gamma_{-, \Phi_-}, k_{\Phi_-}^{\vec{-}}, l_{\Phi_-}^{\vec{-}}\rangle. \quad (2.13)$$

By going to a graph finer than $\gamma_{+, \Phi_+}, \gamma_{-, \Phi_-}$ the right hand side can again be written as a chargenet labelled by a single graph with each edge labelled by a set of 4 charges namely the ‘+’ and ‘-’ embedding and matter charge labels.

3. Group Averaging

Physical states can be constructed by group averaging [11] over the action of all Φ_+, Φ_- . From [10] we have that the group average of any charge net $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle := |\gamma_+, k^{\vec{+}}, l^{\vec{+}}\rangle \otimes |\gamma_-, k^{\vec{-}}, l^{\vec{-}}\rangle$ is the distribution Ψ given by:

$$\begin{aligned} \Psi &:= \sum_{(\Phi_+, \Phi_-) \in \text{Orbit}} \langle \gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}} | \hat{U}_+^\dagger(\Phi_+) \hat{U}_-^\dagger(\Phi_-) \\ &= \sum_{(\Phi_+, \Phi_-) \in \text{Orbit}} \langle \gamma_{+, \Phi_+}, k_{\Phi_+}^{\vec{+}}, l_{\Phi_+}^{\vec{+}} | \otimes \langle \gamma_{-, \Phi_-}, k_{\Phi_-}^{\vec{-}}, l_{\Phi_-}^{\vec{-}} | \end{aligned} \quad (2.14)$$

where Orbit comprises of gauge transformations such that for each distinct chargenet which is gauge related to $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ there is a unique element (Φ_+, Φ_-) in Orbit which maps $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ to this gauge related image. In other words, the sum is over all distinct gauge related chargenets. We shall ignore issues of ambiguities in the group averaging procedure (see for example [9, 10, 17]) as this will not be important for the arguments in this paper.

⁵ While the matter charges are unaffected, each time an edge moves past the point $x = 0 \equiv x = 2\pi$ its embedding charge labels are augmented by factors of L where L is the length of the $T = \text{constant}$ circles in flat spacetime. This is one of the subtleties arising from circular spatial topology.

III. THE FINEST LATTICE SECTOR AND ITS PICTORIAL REPRESENTATION

Since the embedding charges are *eigen values* of the embedding coordinate operators, we can associate an embedding charge net with a discrete slice of Minkowski spacetime as follows. For every edge e the pair $\hbar k_e^+, \hbar k_e^-$ specifies the point $X^+ = \hbar k_e^+, X^- = \hbar k_e^-$ in flat spacetime. The set of such points for all edgewise pairs of eigen values then defines a set of points in the flat spacetime which we may refer to as a discrete slice. It turns out [10, 13] that there exists a superselected sector (with respect to all gauge transformations together with a complete set of Dirac observables) of states with the following ‘finest lattice’ property.

Consider a lightcone lattice of spacing a in the flat spacetime so that $X^+, X^- \in \mathbf{Z}a$ on this lattice. We shall say that a pair of points are nearest neighbours if they are either light like separated and one lattice spacing away from each other or if they are spacelike separated and located at a spatial distance a (as measured by the flat spacetime metric) from each other. Thus each point on the lattice has 6 nearest neighbours, 4 lightlike and 2 spacelike.⁶ Next, consider the set of edgewise pairs of embedding charges for any charge net in the superselected sector under discussion. Plot these as a set of points in flat spacetime in the manner described above. Then it turns out that these points fall on the spacetime lattice $X^+, X^- \in \mathbf{Z}a$ in such a way that points obtained from adjacent edges are nearest neighbours. Further, by virtue of the minimal spacing of the eigen values of the X^\pm operators (2.8)- (2.9), flat spacetime point sets defined by *any* chargenet in the kinematic Hilbert space cannot fall on any finer lattice. Finally, for any charge net in this sector (see (i) below), the matter charges are distributed on the underlying graph in a coarser manner than the embedding charges. Hence with each pair of embedding charges which specify a point in flat spacetime, we can *uniquely* associate a pair of matter charges. This means that we can label each point on the discrete slice in flat spacetime by a pair of matter charges so that the chargenet can be interpreted as a specification of quantum matter on a discrete ‘Cauchy’ slice, this slice satisfying the ‘nearest neighbour’ property on the finest lattice allowed by the spectrum (2.8),(2.9). No other chargenet outside this superselection sector admits this interpretation. Hence this sector is called the ‘finest lattice’ sector.

Technically, chargenets in this sector are specified as follows [13]. A chargenet $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle = |\gamma_+, \vec{k}^+, \vec{l}^+\rangle \otimes |\gamma_-, \vec{k}^-, \vec{l}^-\rangle$ belongs to the finest lattice sector iff (i)- (iii) below hold:

(i) its matter chargenet labels are ‘coarser’ than the embedding ones so that each pair of successive edges of the coarsest graph γ_\pm^{coarse} underlying $|\gamma_\pm, \vec{k}^\pm, \vec{l}^\pm\rangle$ is necessarily labelled by distinct pairs of \pm embedding charges but not necessarily distinct pairs of \pm matter charges.

(ii) its embedding charges on the coarsest graph γ_\pm^{coarse} underlying $|\gamma_\pm, \vec{k}^\pm, \vec{l}^\pm\rangle$ satisfy $\hbar k_{e'_\pm}^\pm - \hbar k_{e_\pm}^\pm = a$ where e'_\pm, e_\pm are adjacent edges in γ_\pm^{coarse} such that e'_\pm lies to the right of e_\pm in the coordinatization x (i.e. the edges are located such that given any point p_\pm in the

⁶ There are also 2 ‘timelike’ neighbours. It turns out that these are not relevant to our discussion. Hence we exclude them from our definition of nearest neighbours.

interior of e_{\pm} with coordinate $x = x_{\pm}$ and any point p'_{\pm} in the interior of e'_{\pm} with coordinate $x = x'_{\pm}$, we have that $x'_{\pm} > x_{\pm}$).

Further, if the difference in the embedding charge value on the last edge and the first edge of γ_{\pm}^{coarse} is $\pm \frac{L}{h}$, the matter charge values on these edges are identical.⁷

(iii) there are N_+ ‘+’ and N_- ‘-’ distinct embedding charges with N_+ such that either $N_+ = N + 1$ or $N_+ = N$ and N_- such that either $N_- = N + 1$ or $N_- = N$ where N is defined in equation (2.10).⁸

Property (i) ensures that we may think of the matter charges l_e^+, l_e^- as sitting on the lattice point $\hbar k_e^+, \hbar k_e^-$. Property (ii) is the technical formulation of the ‘nearest neighbour’ condition. Property (iii) turns out to be necessary for the consistency of the ‘discrete Cauchy slice’ interpretation in the context of circular spatial topology. To summarise: The kinematic charge nets of polymer PFT in this sector can be interpreted as describing quantum matter degrees of freedom on discrete Cauchy slices which fall on a (light cone) lattice in Minkowski spacetime. For the remainder of this work we shall focus, for concreteness, exclusively on states in this finest lattice sector.

Let us consider the action of a gauge transformation Φ_+ on a finest lattice state $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$. It moves the ‘+’ charged edges along the circle relative to the fixed ‘-’ charged edges. Consider a fine enough graph which underlies the new set of ‘+’ edges together with the old ‘-’ edges. On this graph the set of *edgewise pairs* of charges is different from the set corresponding to the original charge net. This new set defines a *new* discrete Cauchy slice and matter data on this slice. Thus, the matter data *propagate* from one discrete Cauchy slice to another. A similar picture ensues for the action of a gauge transformation Φ_- and one explicitly sees how, just as in classical theory, the finite gauge transformations generated by the constraints propagate matter from one slice to another. It turns out (see [10] for details) that the considerations of section II B 2 in conjunction with Footnote 5 ensure that, for appropriate choices of gauge transformations, the initial and final discrete slices can be ‘macroscopically’ separated (i.e. by arbitrarily many lattice points) in time so that such gauge transformations implement *long range propagation*.

From the discussion in section II B 3 a physical state obtained by group averaging over a charge net $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$ can be written as a sum over all distinct charge net states obtained from this one by action of all possible finite gauge transformations. As asserted in [10], if we plot the lattice points in flat spacetime associated with each of these states together with their matter charge labelling, one obtains a single discrete spacetime lattice with uniquely specified matter charges at each lattice point. This specification is *consistent* in the sense that if a single lattice point derives from the same pair of embedding charges arising from different states in the sum, the matter charge labels for this point provided by these different states are the same. In other words the discrete Cauchy slices with matter data which occur as summands in (2.14) stack up consistently to give a single discrete spacetime with quantum matter at each spacetime point. Putting this picture together with that of the action of finite gauge transformations discussed above it follows that any physical

⁷ This additional restriction on the matter charges is due to subtleties connected with circular spatial topology (see Footnote 5).

⁸ We assume that $N \gg 4$ for certain technical reasons (related to the circular spatial topology) in connection with proofs in the following sections and in the Appendix.

state corresponding to a group averaged finest lattice charge net encodes quantum matter propagation on a discrete spacetime.

Before we conclude this section, we define some useful terminology. First note that *edges* of a chargenets define *points* in flat spacetime. If two successive edges define light like related points we refer to the vertex of the chargenets at which these edges intersect as a *null* vertex. If the points are spacelike related we refer to the corresponding vertex as a *spacelike* vertex. Note that from (ii) above successive edges can never define flat spacetime points which are timelike separated.

IV. ULTRALOCALITY OF THE HAMILTONIAN CONSTRAINT AND SMOLIN'S CRITICISM

In the section IIB3 we showed that physical states could be constructed by group averaging over the finite transformations generated by the 'light cone' constraints H_+, H_- . Since Smolin's criticisms apply to the LQG formalism wherein finite spatial diffeomorphisms are averaged over and the Hamiltonian constraint is constructed via Thiemann's procedure, we now turn our attention to a similar treatment for PFT in terms of its diffeomorphism and Hamiltonian constraints, $C_{diff}(2.5)$ and $C_{ham}(2.6)$

First consider, as in LQG, the finite transformations generated by C_{diff} . Recall from the discussion after (2.5), (2.6) that these transformations correspond to the case $\Phi_+ = \Phi_-$. Indeed, setting $\Phi_+ = \Phi_- = \Phi$ in equation (2.13) we see that the associated unitary transformations $\hat{U}_+(\Phi)\hat{U}_-(\Phi)$ drag *both* the '+' and '-' chargenets by the same diffeomorphism Φ , which of course, just corresponds to the action of a spatial diffeomorphism ⁹:

$$\begin{aligned}\hat{U}_{diff}(\Phi)|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle &= \hat{U}_+(\Phi)\hat{U}_-(\Phi)|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle \\ &= |\gamma_{+, \Phi}, \vec{k}_{\Phi}^+, \vec{l}_{\Phi}^+\rangle \otimes |\gamma_{-, \Phi}, \vec{k}_{\Phi}^-, \vec{l}_{\Phi}^-\rangle.\end{aligned}\quad (4.1)$$

By going to a finer graph that $\gamma_{+, \Phi}, \gamma_{-, \Phi}$, one can denote the tensor product chargenets in the last line of the above equation by a single chargenets with edges labelled by a quadruple of (2 embedding and 2 matter) charges. Since both the '+' and '-' labels are dragged around by the same diffeomorphism, it is straightforward to check that $\hat{U}_{diff}(\Phi)|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$ and $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$ define the same discrete Cauchy slice in flat spacetime with the same matter data. This is as it should be because the flat spacetime picture encodes the *relation* between the embedding and matter excitations and this relation is diffeomorphism invariant. Averaging over diffeomorphisms can then be done [13]. Specialising (2.14) to the case $\Phi_+ = \Phi_- = \Phi$, and ignoring group averaging ambiguities, we have that

$$\begin{aligned}\Psi_{diff} &:= \sum_{\Phi \in \text{Orbit}_{diff}} \langle \gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^- | \hat{U}_+^\dagger(\Phi) \hat{U}_-^\dagger(\Phi) \\ &= \sum_{\Phi \in \text{Orbit}_{diff}} \langle \gamma_{+, \Phi}, \vec{k}_{\Phi}^+, \vec{l}_{\Phi}^+ | \otimes \langle \gamma_{-, \Phi}, \vec{k}_{\Phi}^-, \vec{l}_{\Phi}^- | \end{aligned}\quad (4.2)$$

where Orbit_{diff} consists of elements which uniquely take the charge net being averaged over

⁹ This turns out to be true despite the added subtleties alluded to in Footnote 5 [9, 10].

to its distinct diffeomorphic images. Thus the distribution Ψ_{diff} is a sum over distinct diffeomorphic images of the chargenet being averaged so that if a certain bra is in the sum, then all its diffeomorphic images are also in the sum.

Next consider the Hamiltonian constraint C_{ham} (2.6). Recall from discussion after (2.5)-(2.6) that the finite transformations generated by C_{ham} correspond to the case $\Phi_+ = (\Phi_-)^{-1}$. From Reference [13], it follows that with particular quantization choices¹⁰ at finite triangulation, the finite triangulation approximant to the Hamiltonian constraint $C_{ham}(N)$ smeared with lapse N can be written as

$$\hat{C}_{ham,\delta}(N)|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle = -i\hbar \sum_v N(v) \frac{\hat{U}_+(\Phi_{\delta,v})\hat{U}_-(\Phi_{\delta,v}^{-1}) - \mathbf{1}}{\delta} |\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle \quad (4.3)$$

$$= -i\hbar \sum_v N(v) |\gamma_{+, \Phi_\delta}, \vec{k}_{\Phi_\delta}^+, \vec{l}_{\Phi_\delta}^+\rangle \otimes |\gamma_{-, \Phi_\delta^{-1}}, \vec{k}_{\Phi_\delta^{-1}}^-, \vec{l}_{\Phi_\delta^{-1}}^-\rangle \quad (4.4)$$

The sum in (4.4) is over ‘non-trivial’ vertices of the (finest lattice) charge net $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$. By a ‘non-trivial’ vertex we mean a point on γ at which the ingoing and outgoing edges carry non-identical charges. Since the charge net is a finest lattice charge net, this implies that at least one of the incoming embedding charges differs from its outgoing counterpart. $\Phi_{\delta,v}$ is a small diffeomorphism around the non-trivial vertex v which moves v to its right by an amount δ as measured by the coordinate x , where ‘right’ means direction of increasing x . In addition $\Phi_{\delta,v}$ is identity outside a region of size of order δ around v and $\Phi_{\delta,v}$ is such that its inverse $\Phi_{\delta,v}^{-1}$ moves v to its left by an amount δ . Thus given a chargenet, the Hamiltonian constraint acts only at its non-trivial vertices and moves the ‘+’ part of the charge net in the vicinity of the vertex v to the right and the ‘-’ part of the chargenet in the vicinity of the vertex v to the left. For small enough δ it is easy to see that the diffeomorphism class of the chargenet deformed in this manner remains the same as $\delta \rightarrow 0$. Finally note that we may extend the sum in (4.4) to include any number of ‘trivial’ vertices of γ for which the incoming and outgoing charges are identical because, as is straightforward to verify, on any such vertex v , the operator $\hat{U}_+(\Phi_{\delta,v})\hat{U}_-(\Phi_{\delta,v}^{-1})$ acts as the identity on the chargenet so that these vertices do not yield non-zero contributions. It is convenient to define:

$$\hat{U}_{ham,\delta,v} := \hat{U}_+(\Phi_{\delta,v})\hat{U}_-(\Phi_{\delta,v}^{-1}). \quad (4.5)$$

so that

$$\hat{C}_{ham,\delta}(N)|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle = -i\hbar \sum_v N(v) \frac{\hat{U}_{ham,\delta,v} - \mathbf{1}}{\delta} |\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle \quad (4.6)$$

where it is understood that the sum ranges over all non-trivial vertices but can also include any number of trivial vertices as well.

Next, we note that choices could equally well be made in the Thiemann procedure¹¹ such

¹⁰ We shall comment more on these choices in Section VI.

¹¹ We discuss this further in Section VI.

that we obtain the action of the finite triangulation constraint to be

$$\hat{C}_{ham,\delta}(N)|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle = i\hbar \sum_v N(v) \frac{\hat{U}_{ham,\delta,v}^\dagger - \mathbf{1}}{\delta} |\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle \quad (4.7)$$

where, from equation (4.5), we have that

$$\hat{U}_{ham,\delta,v}^\dagger = \hat{U}_-^\dagger(\Phi_{\delta,v}^{-1})\hat{U}_+^\dagger(\Phi_{\delta,v}) = \hat{U}_-(\Phi_{\delta,v})\hat{U}_+(\Phi_{\delta,v}^{-1}) = \hat{U}_+(\Phi_{\delta,v}^{-1})\hat{U}_-(\Phi_{\delta,v}) \quad (4.8)$$

The action of the finite triangulation Hamiltonian constraint (4.7) would then be to move the ‘+’ part of the charge net in the vicinity of the vertex v to the *left* and the ‘-’ part of the charginet in the vicinity of the vertex to the *right*. Clearly the actions (4.6), (4.7) are ultralocal in the sense of Smolin. More in detail, these actions are only in the vicinity of vertices of the charginet being acted upon and the action at one vertex is completely independent of the other. We now analyse repeated actions of the Hamiltonian constraint and show that they do not lead to propagation.

We shall focus first on the action of $\hat{U}_{ham,\delta,v}$ at a vertex v with incoming embedding charges (k_1^+, k_1^-) and outgoing charges (k_2^+, k_2^-) . The action bifurcates the vertex, giving rise to two new vertices v_1, v_2 with v_1 to the left of v_2 and one new edge connecting these vertices so that the sequence of charges from left to right is now $(k_1^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-)$. Thus the sequence of charges in the vicinity of v changes as:

$$(k_1^+, k_1^-), (k_2^+, k_2^-) \rightarrow (k_1^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-) \quad (4.9)$$

The following three cases are of interest (see the end of section III for the terminology used in (ii),(iii) below):

- (i) The original vertex v is trivial: In this case the charge net is unchanged.
- (ii) The vertex v is *null* so that $k_1^+ = k_2^+$ or $k_1^- = k_2^-$: In this case the new charge sequence is equivalent to the one on the original charge net so that the new charginet is diffeomorphic to the old one. It follows that the new charge net defines exactly the same points in flat spacetime as the old charge net.
- (iii) The vertex v is *spacelike* so that $k_1^+ \neq k_2^+$ and $k_1^- \neq k_2^-$. In this case we see that the new vertices v_1 and v_2 are null. Further from (ii), section III and (4.9) it follows that the new charginet represents ‘one lattice step’ of evolution with respect to the original charginet.

Exactly the same conclusions ensue for the action (4.7). Note that the matter charges are just dragged along together with the embedding charges by the actions (4.6),(4.7). From the above discussion we see that if v is trivial or null the Hamiltonian constraint actions (4.6),(4.7) do not change the flat spacetime points (and the matter data thereon) associated with the charginet so that there is no evolution in the flat spacetime. If v is spacelike, it follows from (iii) above that the action of the Hamiltonian constraint (4.6),(4.7) evolves matter data by one lattice step. Note however that this action replaces the spacelike vertex v of the charge net by a pair of null vertices v_1 and v_2 . From (ii) above it follows that *further* actions of the Hamiltonian constraint (whether (4.6) or (4.7)) at these new vertices of the charginet do not evolve the discrete Cauchy slice any further.

Applying these results to all the vertices of the charginet we conclude that repeated actions of the Hamiltonian constraint (say n_1 actions (4.6) followed by m_1 actions (4.7) followed by n_2 of (4.6) and so on) cannot evolve the discrete Cauchy slice with quantum

matter any further than one lattice step away in flat spacetime. It is in this precise sense that the Hamiltonian constraint does not generate long range propagation and it is in this precise sense that Smolin's criticism is formulated in the case of PFT.

In order to understand exactly why repeated actions of the Hamiltonian constraint fail to generate long range propagation, it is appropriate to compare the actions (4.6), (4.7) with that of *finite* gauge transformations which, as explained in section III and [10], *do* describe long range propagation. Let us consider 3 successive edges e_3, e_1, e_2 of the charge net so that the embedding charge sequence is now $(k_3^+, k_3^-), (k_1^+, k_1^-), (k_2^+, k_2^-)$. Let us call the successive vertices u and v so that u is the '3-1' vertex and v , as before, is the '1-2' vertex and let us assume that these vertices are spacelike. Now let us consider a sequence of actions of '-' diffeomorphisms each of which stretches the '-' labels to the left and each of which is identity to the left of e_3 and to the right of e_2 . In visualising this action recall that the charge network has a '+' \otimes '-' product structure (2.11). With an appropriate choice of sequence of such '-' diffeomorphisms, we obtain the following sequence of embedding charges:

$$(k_3^+, k_3^-), (k_1^+, k_1^-), (k_2^+, k_2^-) \rightarrow (k_3^+, k_3^-), (k_1^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-) \rightarrow (4.10)$$

$$(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-) \rightarrow (k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-) (4.11)$$

$$\rightarrow (k_3^+, k_3^-), (k_3^+, k_1^-), (k_3^+, k_2^-), (k_1^+, k_2^-), (k_2^+, k_2^-) (4.12)$$

Thus, at the end of this sequence of actions, the '-' edge with charge ' k_2^- ' has moved leftward so as to overlap the k_3^+ edge yielding the point $(\hbar k_3^+, \hbar k_2^-)$ which is *2 lattice spacings* away from points on the original slice. In order for this to happen in a continuous manner as depicted in (4.10)- (4.12), it is *necessary* to have the intermediate step (4.11) where the k_2^- charge completely displaces the k_1^- charge from the '+' edge with label k_1^+ .

In contrast, since the action of the Hamiltonian constraint (4.6), (4.7) is for sufficiently small δ , this action even if repeated, can never completely erase any of the original pairs of edge labels. In particular even after repeated actions of the Hamiltonian constraint, there is always an edge with the label (k_1^+, k_1^-) . This survival of the original edge labels is directly tied to the ultralocality of the action of the Hamiltonian constraint: since its action only modifies the structure in a small enough¹² neighbourhood of a vertex, each one of the original edges always has a part which is not affected by this action. Viewed in this manner, it is the *inability* of repeated actions of the Hamiltonian constraint to erase such original pairs of labels which obstructs long range propagation.

As mentioned in section I, in the next section we offer a new perspective on propagation and show how this obstruction is sidestepped. Once this obstruction is sidestepped it is not difficult to prove that long range propagation ensues. Since we believe that this proof is just an added layer of polymer PFT technicalities, we relegate this proof to the appendix and concentrate in the main body of the paper on the key lesson of this paper, namely the evasion of the obstacle described above and the robust structural reasons for this evasion.

V. PROPAGATION

As shown in the last section and as anticipated by Smolin, repeated ultralocal actions of the finite triangulation Hamiltonian constraint on a kinematic state do not propagate quan-

¹² The precise notion of 'small enough' is defined in **L1**, section V

tum excitations over long ranges. In this section we reformulate the notion of propagation in terms of properties of physical states which lie in the joint kernel of the diffeomorphism and Hamiltonian constraints.

Recall that solutions to the diffeomorphism constraint (see equation (4.2)) do not lie in the kinematic Hilbert space. Instead they are kinematically non-normalizable *distributions* which may be expressed as formal *sums* over kinematic states. Hence a putative solution to both the Hamiltonian and the diffeomorphism constraints must also be a distributional sum. Consider such a solution and let a finest lattice charge net describing some discrete Cauchy slice with quantum matter be a summand in the sum which represents this solution. Recall, from section III that this slice (together with its quantum matter) evolves under the action of the finite gauge transformations (Φ_+, Φ_-) . If the finest lattice charge net corresponding to *any* finite evolution of this discrete slice with quantum matter is also a summand in the sum which represents this solution then we shall say that the solution encodes propagation effects.¹³

In the previous section we isolated the key obstruction to long range evolution by repeated actions of the Hamiltonian constraint. We now show how our new formulation of propagation overcomes this obstruction. Specifically, as in section IV, let us consider 3 successive edges of a finest lattice charge net with embedding charge sequence $(k_3^+, k_3^-), (k_1^+, k_1^-), (k_2^+, k_2^-)$ with successive vertices u and v so that u is the ‘3-1’ vertex and v is the ‘1-2’ vertex. Recall from section IV that repeated actions of the finite triangulation Hamiltonian constraint on this charge net are unable to produce the charge net with embedding charge sequence $(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-)$ (see equation (4.11)), this inability being the key obstruction to the generation of long range evolution through such repeated actions. We now show that if a physical state in the kernel of the diffeomorphism and Hamiltonian constraints has the original ‘ $(k_3^+, k_3^-), (k_1^+, k_1^-), (k_2^+, k_2^-)$ ’ chargenet as a summand, it *necessarily* has a chargenet with the desired $(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-)$ sequence. We proceed as follows.

Let Ψ be a distribution represented by a sum of ‘bra’ states. We shall say that a chargenet $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$ is in Ψ iff the bra $\langle \gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^- |$ is a summand in the sum over bras which represents Ψ . Next recall that Ψ is a solution to the continuum limit of the finite triangulation Hamiltonian constraint (4.6) iff for every $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$ we have that

$$\lim_{\delta \rightarrow 0} \Psi(-i\hbar \sum_v N(v) \frac{\hat{U}_{ham,\delta,v} - \mathbf{1}}{\delta} |\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle) = 0. \quad (5.1)$$

Now suppose that Ψ is a solution to the Hamiltonian constraint (5.1) and that $|\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$ is in Ψ . Then it must be the case that for all sufficiently small δ that $\hat{U}_{ham,\delta,v} |\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-\rangle$ is also in Ψ (else the nontrivial contribution from the ‘ $\mathbf{1}$ ’ term in (5.1) will not be cancelled). A similar argumentation leads to the converse namely that

¹³ As seen in section III this statement of propagation holds for physical states obtained by group averaging. So we could as well have formulated propagation as the condition that the kernel of the diffeomorphism and Hamiltonian constraints be identical to the physical state space obtained by group averaging. We choose to formulate the statement in the way we have done because such a formulation generalises more easily to the case of LQG where we do not have the possibility of group averaging over the transformations generated by *all* the constraints.

if Ψ satisfies (5.1) and if $\hat{U}_{ham,\delta,v}|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ for all sufficiently small δ , then $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is also in Ψ .

Similarly, Ψ is a solution to the continuum limit of the finite triangulation Hamiltonian constraint (4.7) iff for every $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ we have that

$$\lim_{\delta \rightarrow 0} \Psi(i\hbar \sum_v N(v) \frac{\hat{U}_{ham,\delta,v}^\dagger - \mathbf{1}}{\delta} |\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle) = 0 \quad (5.2)$$

Similar arguments imply that for Ψ which is a solution to the Hamiltonian constraint (5.2), iff $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ then for all sufficiently small δ , $\hat{U}_{ham,\delta,v}^\dagger |\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ .

It is also useful for what follows to recall the following:

- (a) the action of $\hat{U}_{ham,\delta,v}$ on any charge net $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is such that for small enough $\delta > 0$ the charginets $\hat{U}_{ham,\delta,v}|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ are in the *same* diffeomorphism class,
- (b) a similar statement holds for $\hat{U}_{ham,\delta,v}^\dagger$,
- (c) Ψ is a solution to the diffeomorphism constraints iff it is a linear combination of states of the form (4.2).

From the above discussion, it is straightforward to check that the following Lemma **L1** holds.

L1: Let Ψ be a solution to the diffeomorphism constraint, the Hamiltonian constraint in (5.1) and the Hamiltonian constraint in (5.2). Then the following statements hold:

- (i) The charge net $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ iff all diffeomorphic images of $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ are in Ψ ,
- (ii) The charge net $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ iff for all sufficiently small $\delta > 0$, $\hat{U}_{ham,\delta,v}|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ ,
- (iii) The charge net $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ iff for all sufficiently small $\delta > 0$, $\hat{U}_{ham,\delta,v}^\dagger |\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ is in Ψ ,

where in (ii) (respectively (iii)) ‘sufficiently small’ means ‘sufficiently small that the diffeomorphism class of $\hat{U}_{ham,\delta,v}|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ (respectively $\hat{U}_{ham,\delta,v}^\dagger |\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$) remains the same for all such δ .¹⁴

We are now ready to state and prove our desired result.

Proposition: Let Ψ be a solution to the diffeomorphism constraint, the Hamiltonian constraint (5.1) and the Hamiltonian constraint (5.2). Let $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ be a finest lattice charginet in Ψ . Let $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ have 3 consecutive edges¹⁵ with embedding

¹⁴ This characterization of ‘sufficiently small’ follows from (a)-(c) and the fact that Ψ in **L1** is, in particular, a solution to the diffeomorphism constraint.

¹⁵ The proof below applies independent whether the 2 vertices associated with these edges are spacelike or

charges $(k_3^+, k_3^-), (k_1^+, k_1^-), (k_2^+, k_2^-)$. Then the chargenet with these 3 edges replaced by 4 successive edges with the embedding charge sequence $(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-)$ is *necessarily* in Ψ .

Proof: Denote the ‘1-2’ vertex of $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ by u and the ‘2-3’ vertex by v . There are 3 steps to the proof:

Step 1: Act with $\hat{U}_{ham, \delta_1, v}$ on $|\gamma, k^{\vec{+}}, k^{\vec{-}}, l^{\vec{+}}, l^{\vec{-}}\rangle$ at its ‘1-2’ vertex v . We obtain the chargenet with the above sequence replaced by

$$(k_3^+, k_3^-), (k_1^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-). \quad (5.3)$$

L1(ii) implies the chargenet so obtained is also in Ψ (we have implicitly chosen δ_1 small enough that **L1(ii)** applies).

Step 2: Act on the chargenet obtained at the end of Step 1 with $\hat{U}_{ham, \delta_2, u}$ on its ‘3-1’ vertex u to get the sequence

$$(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-) \quad (5.4)$$

L1 (ii) implies the chargenet so obtained is also in Ψ (similar to Step 1, we have implicitly chosen δ_2 small enough that **L1(ii)** applies).

Step 3: Consider the (desired) chargenet with sequence

$$(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-) \quad (5.5)$$

and call the vertex at the intersection of the (k_3^+, k_1^-) and (k_1^+, k_2^-) edges as w . Act on this chargenet by $\hat{U}_{ham, \delta, w}^\dagger$ for sufficiently small δ in the sense of (b) above. It follows that for every such $\delta > 0$, this action yields a chargenet with the sequence (5.4) which is diffeomorphic to the particular chargenet obtained at the end of Step 2. Since the latter chargenet is in Ψ , **L1(i)** implies that the chargenets, obtained by the action of $\hat{U}_{ham, \delta, w}^\dagger$ for all sufficiently small δ on the desired chargenet with sequence (5.5), are in Ψ . The (converse of) **L1(iii)** then implies that the desired chargenet is also in Ψ . This completes the proof.

Note that the first two steps of the proof involve actions by the Hamiltonian constraint on the chargenet in question. Hence, from section IV these steps by themselves are incapable of generating the desired result. It is Step 3 which is the *key* step. The success of this step hinges on the imposition, as a constraint, of the ‘kinematic adjoint’ (4.7),(5.2) of the constraint (4.6),(5.1) together with its ‘ $\hat{U}^\dagger - 1$ ’ structure.

Finally, we also note that, acting by $\hat{U}_{ham, \delta, w}$ on the chargenet (5.5) with sequence

$$(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-) \quad (5.6)$$

at its vertex w (where the (k_3^+, k_1^-) and (k_1^+, k_2^-) edges meet) we obtain a chargenet with sequence

$$(k_3^+, k_3^-), (k_3^+, k_1^-), (k_3^+, k_2^-), (k_1^+, k_2^-), (k_2^+, k_2^-). \quad (5.7)$$

If in the original chargenet the vertices u, v are spacelike, it is easy to see that the point

not.

$(\hbar k_3^+, \hbar k_2^-)$ represents a 2 lattice displacement from the original set of points corresponding to $(\hbar k_3^+, \hbar k_3^-)$, $(\hbar k_1^+, \hbar k_1^-)$, $(\hbar k_2^+, \hbar k_2^-)$. Thus as indicated in section I we are able to demonstrate evolution beyond 1 lattice displacement to 2 lattice displacements. As mentioned at the end of section IV, the demonstration of long range evolution is relegated to the Appendix.

VI. DISCUSSION

Before we proceed to more general remarks, we comment on the derivation of the constraint action (4.7) from choices of finite triangulation approximants to the local fields which comprise the Hamiltonian constraint. The reader not interested in fine technical details of our prior work [13] may skip the next two paragraphs and go on to peruse the more general remarks.

First consider the derivation of the action (4.6). In this regard Reference [13] provides a detailed derivation of finite triangulation approximants to H_+ , H_- and it is from these that the approximant to the Hamiltonian constraint operator (equation (112), [13]) is obtained. It is straightforward to see that this is the same as (4.6) albeit in a slightly different notation. Recall that in order to obtain the desired constraint operator action (112), [13], the embedding sector approximants to the local fields in the constraint are constructed straightforwardly in [13], first as appropriate classical approximants involving state dependent charges, and then as operators. Also recall that Reference [13] is unable to derive matter sector approximants in this way. More in detail, that work is unable to construct classical approximants to the local matter fields in the Hamiltonian constraint such that their replacement by operators is consistent with the desired action (112) of [13]¹⁶. Hence an indirect appeal to the Hamiltonian vector fields of the matter part of the constraints is made [13] and this constitutes a slight departure from the strict ‘Thiemann-like’ prescription.

Next consider the derivation of the adjoint action (4.7). It turns out that the action (4.7) requires a slightly different choice of approximants to local fields in the constraints. It is once again straightforward to construct the desired embedding momenta approximants¹⁷. However, for the matter sector one needs to again consider Hamiltonian vector fields. While we have not done this in detail, we anticipate that the considerations of Section VB, [13] may be mimicked with slightly different choices so as to obtain an action which contributes appropriately to the Hamiltonian constraint approximant so as to obtain (4.7). More generally, our current viewpoint is that it is the Hamiltonian vector field structure of the constraint rather than the constraint itself which is primary and that this Hamiltonian vector field structure is what one should import in a suitable fashion into quantum theory even if one is unable to provide concrete classical approximants to the constraint itself. From this viewpoint one can directly posit the actions (4.6), (4.7) as approximants to the infinitesimal transformations generated by the constraint (2.6) without unduly worrying about classical approximants to the constraint itself.

Finally, in this work we have *not* imposed the zero mode constraint [9, 10] as we feel

¹⁶ We think that the underlying reason for this inability is that, as mentioned in IIB 1, there is no ‘unpolymerised’ matter variable.

¹⁷ See the discussion immediately after (47), [13]. There is a typographical error in the choice of approximant for the ‘+’ embedding momenta which is claimed to lead to the leftward displacement of the ‘+’ vertex: the subscript on the embedding holonomy should be $\Delta - 1$ instead of Δ .

that it does not impinge on the issues we are concerned with here. For completeness, this constraint should be imposed (else the classical arena is not a phase space [9, 10]). Since this constraint commutes with the action of finite gauge transformations, it commutes with the action of finite spatial diffeomorphisms, and the action of the finite triangulation Hamiltonian constraint (4.6), (4.7). Since it commutes with the latter at any value of the triangulation parameter δ , we expect it to commute with the continuum limit action of (4.6), (4.7). Hence we expect that it shouldn't matter whether we find the kernel of the Hamiltonian and diffeomorphism constraints first and then group average these solutions over the zero mode constraint or whether we first solve the zero mode constraint by group averaging, define the Hamiltonian and diffeomorphism constraints on the resulting states and then find the kernel of the Hamiltonian and diffeomorphism constraints. Verifying the above expectations, while straightforward, lies outside the scope of this work. Incidentally, we believe that in hindsight, the treatment of the zero mode in [13] was too perfunctory, that the arguments there should be seen as arguments prior to the implementation of the zero mode constraint and that a proper treatment of the zero mode constraint should explicitly verify our expectations as stated above. This concludes our discussion of fine technical matters in polymer PFT.

We now discuss the key structural lessons from this work for LQG. Let us refer to the new charge nets obtained by the action of a constraint on a given chargenet as ‘children’ of this ‘parent’ chargenet. In this language Smolin’s general considerations imply, correctly, that such children and their descendants do not encode long range propagation. However, given the structure of the constraints (4.6)–(4.7), Lemma **L1** implies that if a parent is in Ψ so are its children, and, conversely if any child is in Ψ then so are all its parents. It is the converse statement which provides the key ingredient of ‘non-unique parentage’ in the crucial Step 3 of our proof in the previous section.

From a general point of view what Step 3 effectively achieves is the merging of two vertices of a ‘child’ into a single vertex of a ‘parent’ (the single vertex being w in our proof). Note that this merging cannot be achieved by the action (4.6) nor by its kinematic adjoint action (4.7) on the child. This is apparent from the considerations of section III which apply equally to both actions, the key point being that these actions are only defined as finite triangulation constraint actions *for sufficiently small δ* . It is this caveat of “*for sufficiently small δ* ” that leads to ultralocality and the impossibility of merging vertices of the child by action on the child. Rather, the merging is achieved by *separating* vertices of the parent via the action of the kinematic adjoint and using the structure of the constraint action in terms of the difference of a unitary operator and the identity to conclude that the existence of children in a physical state imply the existence of all their possible ancestors. Once the 2 vertices of the child have been effectively merged through this structure to yield the desired parent, a suitably chosen action of the Hamiltonian constraint (see the last paragraph of V) on this parent creates a *different* child and the sequence ‘parent \rightarrow child \rightarrow different parent \rightarrow different child’ constitutes an evolution path to a final discrete Cauchy slice two lattice spacings away. This sequence may be viewed as the propagation of a perturbation (namely the k_2^- charge together with the associated ‘-’ matter charges) ‘leftward’ along the charge net.

From the above discussion it is apparent that the key structures responsible for propagation are exactly (i) and (ii) of section I and that propagation should be viewed as encoded in the structure of physical states rather than as a property of repeated actions of the finite triangulation Hamiltonian constraint on kinematical states. To conclude, while we do expect

the general Thiemann procedure to yield a Hamiltonian constraint with ultralocal action, we are optimistic that the structural lessons arising out of this work can be imported in a suitable way to LQG so as to restrict the choice of this ultralocal action in such a way that physical states in the kernel of the Hamiltonian (and diffeomorphism) constraints do encode propagation effects.¹⁸

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Appendix A: Proof of long range propagation

In this appendix we assume familiarity with the contents of [9, 10, 13]. We shall also set $\hbar = 1$ by a suitable choice of units.

Recall the following from [10]. The diffeomorphisms of the circle, Φ_{\pm} , admit periodic extensions to the real line, also denoted by Φ_{\pm} . As in [10], denote the set $(\gamma_{\pm}, \vec{k}^{\pm}, \vec{l}^{\pm})$ by s^{\pm} and the corresponding states $|\gamma_{\pm}, \vec{k}^{\pm}, \vec{l}^{\pm}\rangle$ by $|s^{\pm}\rangle$ with $|s^+\rangle \otimes |s^-\rangle := |s^+, s^-\rangle$. s^{\pm} is referred to as a charge network or charge network label and $|s^{\pm}\rangle$ as a charge network state. Whereas s^{\pm} is defined on $[0, 2\pi]$ its extension s_{ext}^{\pm} is defined on the entire real line by periodic extension of the graph γ^{\pm} , its edges and its matter charge labels and quasi-periodic extension (with appropriate augmentation by factors of $\pm L$) of its embedding charge labels. The periodic extensions of Φ_{\pm} have a well defined action on s_{ext}^{\pm} . The state defined by the restriction of the image of this action to the interval $[0, 2\pi]$ coincides with that obtained by the action of $\hat{U}_{\pm}(\Phi_{\pm})$ on $|s^{\pm}\rangle$ [9, 10] so that :

$$\hat{U}_{\pm}(\Phi_{\pm})|s^{\pm}\rangle = |s_{\Phi_{\pm}}^{\pm}\rangle, \quad s_{\Phi_{\pm}}^{\pm} := \Phi_{\pm}(s_{ext}^{\pm})|_{[0, 2\pi]}. \quad (\text{A1})$$

Clearly, the action of Φ_{\pm} on *any* interval of coordinate length 2π determines its action everywhere on the real line by virtue of the periodicity of this action. Similarly the restriction of s_{ext}^{\pm} to any interval of coordinate length 2π determines s_{ext}^{\pm} everywhere on the real line.

Since $\hat{U}_{ham, \delta, v}, \hat{U}_{ham, \delta, v}^{\dagger}$ are constructed from $\hat{U}_{\pm}(\phi_{\delta, v}), \hat{U}_{\pm}^{\dagger}(\phi_{\delta, v})$ it follows that the action of the finite triangulation Hamiltonian constraint on any state $|s^+, s^-\rangle$ is determined by the action of the diffeomorphisms $\Phi_{\pm} = \phi_{\delta, v}$ (and their inverses) on the restriction of s_{ext}^{\pm} to any interval of coordinate length 2π .

More in detail, for some real y , consider the interval $[y, y + 2\pi]$. Φ_{\pm} maps this interval to the interval $[y^{\pm}, y^{\pm} + 2\pi]$ where we have set $\Phi_{\pm}(y) =: y^{\pm}$. Consider the restriction, $s_{ext}^{\pm}|_{[y, y+2\pi]}$ of s_{ext}^{\pm} to the interval $[y, y+2\pi]$. Clearly Φ_{\pm} has a natural action on $s_{ext}^{\pm}|_{[y, y+2\pi]}$ (it maps every edge e of the graph underlying $s_{ext}^{\pm}|_{[y, y+2\pi]}$ into its image $\Phi_{\pm}(e)$ in $[y^{\pm}, y^{\pm} + 2\pi]$, with $\Phi_{\pm}(e)$ being colored by the same charges as e). Denote the resulting charge net on $[y^{\pm}, y^{\pm} + 2\pi]$ by $\Phi_{\pm}(s_{ext}^{\pm}|_{[y, y+2\pi]})$. It is straightforward to check that $\Phi_{\pm}(s_{ext}^{\pm}|_{[y, y+2\pi]})$ is just the restriction of $\Phi_{\pm}(s_{ext}^{\pm})$ to the interval $[y^{\pm}, y^{\pm} + 2\pi]$. It then follows that the extension, $(\Phi_{\pm}(s_{ext}^{\pm}|_{[y, y+2\pi]}))_{ext}$,

¹⁸ See Footnote 3 in this regard.

of the charge net $\Phi_{\pm}(s_{ext}^{\pm}|_{[y,y+2\pi]})$ to the real line is just $\Phi_{\pm}(s_{ext})$. Finally, the restriction of this extension to $[0, 2\pi]$ is the just the charge net $s_{\Phi_{\pm}}^{\pm}$ i.e.

$$(\Phi_{\pm}(s_{ext}^{\pm}|_{[y,y+2\pi]})_{ext})|_{[0,2\pi]} = s_{\Phi_{\pm}}^{\pm}. \quad (\text{A2})$$

Since $\hat{U}_{\pm}(\Phi_{\pm})|s^{\pm}\rangle = |s_{\Phi_{\pm}}^{\pm}\rangle$, the content of this paragraph is just a transcription to mathematical notation of what we said in words in the previous paragraph.

The above discussion implies that **L1**, Section V may be rephrased as follows in the notation used above (this rephrasing, while cumbersome and seemingly roundabout, is useful for our purposes in this appendix).

L2: Let Ψ be a solution to the diffeomorphism constraint, the Hamiltonian constraint in (5.1) and the Hamiltonian constraint in (5.2). Let $|s^+, s^-\rangle$ be a finest lattice chargenet state. The charge net state $|s^+, s^-\rangle$ is in Ψ iff the set of charge net states $\{|t^+, t^-\rangle\}$ is in Ψ where the elements $|t^+, t^-\rangle$ of this set are defined by any of (i)- (iii) below, with y any fixed real number and $v \in [y, y + 2\pi]$:

- (i) $(\phi(t_{ext}^{\pm})|_{[y,y+2\pi]})_{ext}|_{[0,2\pi]} = s^{\pm}$ for any diffeomorphism ϕ .
- (ii) $t^+ = (\phi_{\delta,v}(s_{ext}^+|_{[y,y+2\pi]}))_{ext}|_{[0,2\pi]}$, $t^- = (\phi_{\delta,v}^{-1}(s_{ext}^-|_{[y,y+2\pi]}))_{ext}|_{[0,2\pi]}$ for all sufficiently small δ .
- (iii) $t^+ = (\phi_{\delta,v}^{-1}(s_{ext}^+|_{[y,y+2\pi]}))_{ext}|_{[0,2\pi]}$, $t^- = (\phi_{\delta,v}(s_{ext}^-|_{[y,y+2\pi]}))_{ext}|_{[0,2\pi]}$ for all sufficiently small δ .

where in (ii)-(iii) ‘sufficiently small’ means ‘sufficiently small that the diffeomorphism class of $|t^+, t^-\rangle$ does not change.’¹⁹

Next note that similar to (2.11) we may consider a fine enough graph which underlies both s^+ and s^- and whose edges carry both $+$ and $-$ charges. Let us denote the resulting label set $(\gamma, \vec{k}^+, \vec{k}^-, \vec{l}^+, \vec{l}^-)$ by s and set

$$|s\rangle := |s^+, s^-\rangle. \quad (\text{A3})$$

We may also define the extended label s_{ext} by a periodic extension γ_{ext} of γ , a periodic extension of the matter charge labels and appropriate (quasi)periodic extensions of the embedding charges. Clearly, γ_{ext} constitutes a fine enough graph which underlies both s_{ext}^+ and s_{ext}^- , and s_{ext} accomodates both the $+$ and $-$ charge labels of s_{ext}^+ and s_{ext}^- .

The Proposition of Section V can then be rephrased as:

P1: Let Ψ be a solution to the diffeomorphism constraint, the Hamiltonian constraint (5.1) and the Hamiltonian constraint (5.2). Let $|s\rangle = |s^+, s^-\rangle$ be a finest lattice chargenet in Ψ . Let y be some real number. Let the restriction of s_{ext} to the interval $[y, y + 2\pi]$ be $s_{ext}|_{[y,y+2\pi]}$. Let the graph underlying $s_{ext}|_{[y,y+2\pi]}$ have 3 consecutive edges e_3, e_1, e_2 with

¹⁹ Recall that $\phi_{\delta,v}$ is such that $\phi_{\delta,v}$ moves v to the right by a coordinate distance δ , its inverse $\phi_{\delta,v}^{-1}$ moves v to the left by a distance δ and $\phi_{\delta,v}$ is identity outside an interval of size of order δ about v .

$e_3, e_1, e_2 \subseteq [y, y + 2\pi]$ with embedding charges $(k_3^+, k_3^-), (k_1^+, k_1^-), (k_2^+, k_2^-)$. Consider the charge net state $|s'\rangle = |s'^+, s'^-\rangle$ such that $s'_{ext}|_{[y, y+2\pi]}$ agrees with $s_{ext}|_{[y, y+2\pi]}$ except that these 3 edges of the latter are replaced in the former by 4 successive edges e'_4, e'_3, e'_1, e'_2 with the embedding charge sequence $(k_3^+, k_3^-), (k_3^+, k_1^-), (k_1^+, k_2^-), (k_2^+, k_2^-)$. Then $|s'\rangle$ is *necessarily* in Ψ .

It is straightforward to check that **P1** can be proved along the lines of the proof of the Proposition of section V

Next, let s, y, s', Ψ be as in **P1**. Clearly, there exists a diffeomorphism ϕ which is identity to the right of e_2 and the left of e_3 in $[y, y + 2\pi]$ and which maps s' to some s'' such that in $s''_{ext}|_{[y, y+2\pi]}$ we have that $\phi(e'_2) = e_2$, $\phi(e'_1) = e_1$, $\phi(e'_4 \cup e'_3) = e_3$. Thus $s''_{ext}|_{[y, y+2\pi]}$ agrees with $s_{ext}|_{[y, y+2\pi]}$ outside the interval $\phi(e'_3) \cup e_1$. **L2** (i) then implies that the following corollary to **P1** holds:

C1: Given s, y, s', s'' as above, $|s''\rangle$ is also in Ψ .

With these results in place we now show that if a finest lattice charge net state $|s\rangle$ is in any diffeomorphism invariant solution Ψ to the Hamiltonian constraints (5.1) and (5.2), then all states related to $|s\rangle$ by the action of finite gauge transformations generated by H_+, H_- are also in Ψ . It then follows from the notion of propagation introduced in section V that Ψ encodes long range propagation effects. We proceed by proving the following lemmas **L3-L7**.

In the proof of **L3** below we shall use the following notion of ‘*sequence of ‘-’ embedding charges*’. Let $s = (s^+, s^-)$ be a finest lattice chargenet. Let s_{ext}^- be the extension of s^- and consider the restriction $s_{ext}^-|_{[y, y+2\pi]}$ to some 2π interval $[y, y + 2\pi]$. Consider any fine enough graph underlying the charges on $s_{ext}^-|_{[y, y+2\pi]}$. Let the edges of this graph be $e_I, I = 1, \dots, B$ where e_J is to the right of e_I for $J > I$. Let the ‘-’ embedding charge on e_I be k_I^- . Then the ordered set of ‘-’ charges $(k_1^-, k_2^-, \dots, k_B^-)$ is referred to as the sequence of ‘-’ embedding charges associated with $s_{ext}|_{[y, y+2\pi]}$. The finest lattice property implies that this sequence is non-increasing and that for the coarsest graph underlying $s_{ext}|_{[y, y+2\pi]}$ this sequence is strictly decreasing. Thus, depending on the fineness of the graph, there may be several instances of a number of successive entries in the sequence being identical. In the proof of **L3** below we shall refer to such sequences directly without explicitly constructing the graphs which define them; however it is to be understood that such graphs exist (as the reader may verify, it is straightforward to show their existence).

L3 Let $|s\rangle = |s^+, s^-\rangle$ be a finest lattice state which is in a solution Ψ to the diffeomorphism constraint and the Hamiltonian constraints (5.1) and (5.2). Let $s'^+ = s^+$. Let s'^- be such that the set of its matter charge labels, ordered edgewise from left to right is identical to the corresponding set for s^- and such that the set of its embedding charge labels, ordered from left to right are obtained by decreasing each of the elements of the corresponding set for s^- by ML for some arbitrary positive integer M . Then $|s'\rangle = |s'^+, s'^-\rangle$ is also in Ψ .

Proof: As in Footnote 8, we shall assume $N \gg 4$. Let the coarsest graph underlying s have edges $e_I, I = 1, \dots, A$ with embedding charges (k_I^+, k_I^-) with e_I to the left of e_J for $J > I$. Clearly $A \gg 4$. Also note that the chargenets in **L2**, **P1**, **C1** are related by the action of gauge transformations so that these chargenets are all in the finest lattice sector and respect

properties (i)- (iii), section III. We shall implicitly use this fact repeatedly in what follows.

Consider s_{ext} and let e_0 be the edge (of the coarsest graph underlying s_{ext}) in the interval $[-2\pi, 0]$ starting at some point $y_0 \in [-2\pi, 0]$ and ending at the origin. In the next 3 paragraphs we will repeatedly apply **C1** to appropriately chosen triplets of edges in the interval $[y_0, y_0 + 2\pi]$, this being the interval spanned by $e_I, I = 0, \dots, A - 1$.

C1 applied to the 3 edges e_0, e_1, e_2 implies that we can displace the ‘-’ charge k_1^- on the edge e_1 by extending the coloring k_2^- to e_1 so that e_1, e_2 are colored by k_2^- . Denote the resulting extended charge net by s_{1ext} so that its restriction to $[0, 2\pi]$ is the charge net s_1 . Clearly the ‘-’ embedding charge on the last edge of s_1 (in the interval $[0, 2\pi]$) is now $k_1^- - L$ (see [9, 10] and Footnote 5). Let $e_0^{(1)}$ be the edge in s_{1ext} which ends at $x = 0$. The edge $e_0^{(1)}$ corresponds to the edge $\phi(e'_3)$ and e_0 to $\phi(e'_4 \cup e'_3)$ in the remarks before **C1**. It follows that $e_0^{(1)}$ is contained in e_0 . Hence, denoting the left end point of $e_0^{(1)}$ by $y(1)_0$, we have that $0 > y_0^{(1)} > y_0$.

Iterate this process to spread k_3^- leftwards in s_1 at the cost of k_2^- as follows. First note that the graph which underlies s_1 can be chosen such that its first 3 edges coincide with e_1, e_2, e_3 . From **C1** we can spread k_3^- to e_2 . From **P1** and **C1** note that in the resulting chargenet the edges e_1, e_2, e_3 can still be taken to be the first 3 edges with e_1 colored by k_2^- and e_2, e_3 by k_3^- . Next consider the extension of this chargenet and the edges $e_0^{(1)}, e_1, e_2$ therein. Using **C1** we can move k_3^- from e_2 to e_1 . This yields the desired charge net s_2 with first edge colored by k_3^- and last edge by $k_2^- - L$. Denote the edge in s_{2ext} ending at $x = 0$ by $e_0^{(2)}$ and let its left end point be $y_0^{(2)}$. Here $e_0^{(2)}$ corresponds to $\phi(e'_3)$ and $e_0^{(1)}$ to $\phi(e'_4 \cup e'_3)$ in the remarks before **C1**. It follows that **C1** implies that $0 > y_0^{(2)} > y(1)_0 > y_0$.

Clearly, we can iterate this process such that after $A - 2$ iterations we obtain s_{A-2} with first charge k_{A-1}^- , last charge $k_{A-2}^- - L$ and the edge in $s_{A-2 ext}$ ending at the origin with left endpoint $y_0^{(A-2)}$ such that $0 > y^{(A-2)} > y^{(A-3)} \dots > y_0$.

Next, consider the interval $[y_0^{(A-2)}, y_0^{(A-2)} + 2\pi]$. In this paragraph we shall repeatedly apply **C1** to appropriately chosen triplets of edges in this interval. Note that the sequence of ‘-’ charges in $s_{A-2 ext}$ restricted to this interval reads $(k_{A-2}^-, k_{A-1}^-, k_A^-, k_1^- - L, k_2^- - L, \dots, k_{A-3}^- - L)$. Repeated application of **C1** to appropriate edges in this interval results in the spread of k_A^- leftwards till $x = 0$ resulting in the charge net s_{A-1} . The sequence of charges in $s_{A-1 ext}$ restricted to this interval reads, as before, $(k_{A-2}^-, k_{A-1}^-, k_A^-, k_1^- - L, k_2^- - L, \dots, k_{A-3}^- - L)$ except that now the edge ending at $x = 0$ has charge k_{A-1}^- and the edge starting at $x = 0$ has charge k_A^- . Finally, we repeatedly apply **C1** to appropriate edges in this interval so as to spread $k_1^- - L$ leftwards till the origin to yield the charge net s_A . The sequence of ‘-’ charges in the restriction of $s_{A ext}$ to this interval is unaltered but now the edge starting out from the origin has charge $k_1^- - L$ and the edge ending at the origin has charge k_A^- . This implies that the charge sequence in s_A in the interval $[0, 2\pi]$ reads $(k_1^- - L, k_2^- - L, \dots, k_{A-3}^- - L, k_{A-2}^- - L, k_{A-1}^- - L, k_A^- - L)$.

Finally taking s_A as the initial charge net and repeating the above procedure M times leads to the desired result.

Note N1: Interchanging the role of $\hat{U}_{\delta,v}$ and $\hat{U}_{\delta,v}^\dagger$ in the proof of the Proposition of Section V and in that of **P1**, **C1** results in the rightward movement of the ‘-’ charges. With this modification considerations identical to **L3** lead to the ‘-’ embedding charges being augmented by factors of $+ML$. Similarly, it is straightforward to see that the Proposition of Section V as well as **P1**, **C1** and **L3** above can be modified to reflect leftward and

rightward movement of the ‘+’ charges with augmentation of the ‘+’ charges by factors of $\pm ML$. The fact that we are dealing with finest lattice charge nets satisfying (ii) of section III immediately implies that the matter charges are dragged along with the embedding charges yielding the desired charge configurations.

L4 Let $|s\rangle = |s^+, s^-\rangle$ be a finest lattice state with N^\pm edges as in (iii) of Section III. Let $|s\rangle$ be in a solution Ψ to the diffeomorphism constraint and the Hamiltonian constraints (5.1) and (5.2). Then there exists $|s'\rangle = |s'^+, s'^-\rangle$ in Ψ such that s'^\pm have $N + 1$ distinct embedding charges i.e. the coarsest graphs γ^\pm underlying $s'\pm$ have $N^\pm = N + 1$ edges.

Proof: The proof consist of the following 2 steps.

Step 1: If $N^- = N + 1$ proceed to Step 2. If not then we have $N^- = N$. From (ii), (iii) Section III, this means that the embedding charge sequence on s^- is of the form $k_1^-, k_2^-, \dots, k_N^-$ with $k_1^- - k_N^- = L - a$. Application of $\hat{U}_{\delta, v=0}$, for small enough δ , to $|s\rangle$ drags part of the first edge of s^- leftwards. The rightmost charge of resulting chargenet s_1 is then, by (quasi)periodicity, $k_1^- - L$ and the left most charge is still k_1^- so that now $N^- = N + 1$. By **L2**, the resulting chargenet state $|s_1\rangle$ is in Ψ .

Step 2: In the charge net state $|s_1\rangle$ at the end of Step 1, if $N^+ = N + 1$ we are done. If $N^+ = N$, an application of $\hat{U}_{\delta', v=0}$ for small enough δ' to $|s_1\rangle$ leads to the rightward dragging of part of the first edge of s_1^+ while maintaining $N^- = N + 1$. It is then easy to see that the resulting charge net has $N^+ = N + 1$. Further the resulting charge net is in Ψ by **L2**.

This completes the proof.

L5 Let $|s\rangle, |s'\rangle$ be finest lattice charge nets with $|s'\rangle$ in Ψ' where Ψ' is a solution to the diffeomorphism constraint and the Hamiltonian constraints (5.1), (5.2). Let s^\pm, s'^\pm have $N + 1$ distinct ‘ \pm ’ embedding charges. Then there exist $|t'\rangle$ in Ψ' such that s^\pm, t'^\pm have identical sets of \pm embedding charge labels.

Proof: Let us number the edges (and their embedding charge labels) in s^-, s'^- from 1, ..N+1 as we proceed rightwards on the coarsest graphs underlying these charge nets. Thus the left most charge in s^- is k_1^- . By the finest lattice property there exists a unique charge label $k_m'^-$ in s'^- with the property that $k_m'^-$ is the largest ²⁰ ‘-’ embedding charge in s'^- such that $k_m'^- - k_1^-$ is an integer multiple ‘ M_- ’ times L . If $m = 1$ then we can apply **L3** to construct t^- such that its ‘-’ embedding charge set agrees with that of s^- .

If $m \neq 1$, we apply **C1** repeatedly to s' so as to spread $k_m'^-$ leftwards in a manner identical to that employed in the proof of **L3**. As a result, on the resulting chargenet $k_m'^-$ becomes the first charge. Since $k_m'^- \neq k_{m-1}'^-$, it is easily verified that this charge net has N distinct ‘-’ charges. It is straightforward to verify that an application of $\hat{U}_{\delta, v=0}$ for small enough δ moves the first edge of the resulting chargenet slightly to the left so as to change N^- to $N + 1$ while maintaining the ‘+’ embedding charge set. Call the chargenet so obtained as s'_1 .

Next, we can identify the unique smallest ‘+’ embedding charge on s'_1 which differs from k_1^+ by an integer multiple M_+ of L . If this charge colors the first edge of s'_1 we are done.

²⁰ This rules out $m = N + 1$

If not, we can spread this charge to the left of s' by application of **N1** so that it becomes the first '+' embedding charge on the resulting chargenet. An application of $\hat{U}_{\delta',v=0}$ for small enough δ' on this chargenet with $N^+ = N$ ensures that the resulting chargenet s'_2 has $N^+ = N + 1$ while maintaining $N^- = N + 1$. As a result s'_2 has the same \pm embedding chargesets as s modulo factors of $M_{\pm}L$.

Finally we can apply **L3**, **N1** to obtain the desired charge net t' . From **L1**, **L3**, **N1** it follows that $|t'\rangle$ is in Ψ' .

L6 Let $|s\rangle, |s'\rangle$ be finest lattice charge nets with $|s'\rangle$ in Ψ' where Ψ' is a solution to the diffeomorphism constraint and the Hamiltonian constraints (5.1), (5.2). Let s^{\pm}, s'^{\pm} have $N+1$ ' \pm ' distinct embedding charges and let the sets of these \pm embedding charge labels be identical. Then there exists $|r'\rangle$ in Ψ' such that the embedding charge nets underlying r', s are identical.

Proof: By an appropriate choice of spatial diffeomorphism ϕ such that $\phi = \mathbf{1}$ in a small neighbourhood of $x = 0$ (and hence $x = 2\pi$), we can arrange for $t = s'_\phi$ to be such that its '+' embedding charge net matches with that of s . Consider the coarsest graphs underlying s^-, t^- . Let their first edges be e_1, f_1 . If $f_1 = e_1$, we can proceed to a comparison of the second edges of these charge nets. If f_1 is longer than e_1 we can stretch the second edge f_2 of t_1 (with charge k_2^-) leftwards by repeated applications of **C1** so that on the (coarsest graph underlying the) resulting charge net t_1^- , the first edge is also e_1 . If f_1 is shorter than e_1 then we can stretch f_1 rightwards by repeated applications of **N1** so that in the coarsest graph underlying the resulting chargenet t_1^- , the first edge is again e_1 .²¹

Next, we compare the second edges on s^-, t_1^- . If they are unequal, we can use **C1**, **N1** to stretch the 3rd edge of t_1^- leftward or the second edge rightward so as to generate t_2^- on which the first two edges match those of s^- . Clearly iterations of this procedure ensure that $r'^- := t_{N-1}^-$ agrees with s_{N-1}^- while maintaining $r'^+ = t^+ = s^+$. From **C1**, **N1** it follows that we have constructed the desired $|r'\rangle$ in Ψ' .

L7: Let $|s'\rangle$ be a finest lattice charge net in Ψ' where Ψ' is a solution to the diffeomorphism constraints and the Hamiltonian constraints (5.1), (5.2). Let s be related to s' by a finite gauge transformation Φ_+, Φ_- . Then $|s\rangle$ is also in Ψ' .

Proof: Since s is gauge related to s' , it is also a finest lattice state. The proof of **L4** implies that there exists $|s_1\rangle$ which is obtained by the action of some $\hat{U}_{\delta,v=0}, \hat{U}_{\delta',v=0}$ on $|s\rangle$ for small enough δ, δ' , such that s_1 has $N+1$ '+' and $N+1$ '-' embedding charges. From **L2** it follows that if $|s_1\rangle$ is in any solution to the diffeomorphism constraint and to (5.1), (5.2), then $|s\rangle$ must be in the same solution.

Next, **L4** implies that there exists s'_1 with $N+1$ '+' and $N+1$ '-' embedding charges such that $|s'_1\rangle$ is in Ψ' and **L5** implies that there exists $|s'_2\rangle$ in Ψ' such that s'_2, s_1 have identical sets of embedding charges. From **L6**, there exists $|s'_3\rangle$ in Ψ' such that s'_3, s_1 have identical embedding charge networks.

Next, it is straightforward to check that, as asserted in [10], if two finest lattice states are gauge related and have the same embedding charge networks, they must have the same matter charge networks.²² Since the transformations relating s to s_1 and s' to $s'_i, i = 1, 2, 3$

²¹ Note that in applying **C1**, **N1**. we are free to choose an appropriately fine graph which underlies t .

²² The interested reader may check that this assertion follows from property (i) section III and the gauge

and s to s' are gauge transformations, it follows that $s'_3 = s_1$. From the first paragraph of this proof it follows that $|s\rangle$ is in Ψ' .

To summarise: We have shown that if a finest lattice state is in a solution to the diffeomorphism and Hamiltonian constraints (5.1),(5.2) then all finite gauge transformations of this state are also in that solution. From section V, this implies that such a solution encodes long range propagation.

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